Boundary conditions for φ will become [3]

$$\varphi(0) = \varphi''(0) = \varphi'(\infty) = 0 \qquad (4.2)$$

By symmetry we have, for ψ,

$$\psi''(0) = 0 \qquad (4.3)$$

and at infinity, we adopt one of the following conditions:

$$\psi(\infty) = 0, \quad 2\psi(\infty) + \varphi''(\infty) = 0 \qquad (4.4)$$

We note that Eqs. (4.1) with the condition (4.2) and the first condition of (4.4), coincide with the equations of motion for a submerged stream of a Newtonian viscous fluid ($\psi = 0$). If, on the other hand, the conditions (4.2) and (4.3) together with the second condition of (4.4) are taken, then the solution of the problem on the submerged stream with couple stresses leads to the process of integrating (4.1).

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EQUILIBRIUM FIGURES OF A ROTATING LIQUID CYLINDER

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The equilibrium figures of a homogeneous right cylinder kept together by surface tension forces are considered. As we know, the only equilibrium cylindrical figure in the absence of rotation is a right circular cylinder (this shape corresponds to minimal surface energy). Such a cylinder remains an equilibrium figure with rotation about the axis of symmetry of the normal cross section. However, as will be shown below, new equilibrium figures in the form of right cylinders with nth order ($n = 2, 3, \ldots$) axes of symmetry arise for certain values of the angular velocity.

1. The equation of the equilibrium figure surface. As we know [1], a rotating liquid figure is kept in equilibrium by surface tension forces if the equation

$$\frac{\rho\Omega^2 r^2}{2} + p_0 = \alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + p_1 \tag{1.1}$$

holds on its surface. Here ρ is the density of the liquid, Ω is the angular velocity of rotation of the liquid mass, p_0 is the hydrodynamic pressure at the axis of rotation, p_1 is the external pressure, r is the distance of a point on the surface from the axis of rotation, R_1 and R_2 are the principal radii of curvature of the surface, and α is the coefficient of surface tension.

In the case of equilibrium cylindrical figures with rectilinear generatrices, surface equation (1.1) can be written as

$$4\omega^2 r^2 = \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} + \frac{D}{R} \qquad \left(\omega^2 = \frac{\rho\Omega^2}{8\alpha}, \quad \frac{D}{R} = \frac{p_1 - p_0}{\alpha}\right) \tag{1.2}$$

Here $r = r(\varphi)$ is the equation of the normal cross section of the cylindrical figure in polar coordinates; the primes denote derivatives with respect to φ.

To Eq. (1.2) we must add the following conditions:

the condition of incompressibility of the liquid,

$$\int_0^{2\pi} r^2\, d\varphi = 2\pi R^2 = \text{const} \tag{1.3}$$

the condition that the centers of mass of the normal cross sections of the liquid figure lie on the axis of rotation,

$$\int_0^{2\pi} r^3 \cos\varphi\, d\varphi = 0, \qquad \int_0^{2\pi} r^3 \sin\varphi\, d\varphi = 0 \tag{1.4}$$

and the condition of single-valuedness,

$$r(\varphi + 2\pi) = r(\varphi) \tag{1.5}$$

2. The solution of Equation (1.2). The bifurcation points. The solution of Eq. (1.2) which satisfies requirements (1.3)–(1.5) is a right cylinder with a circular cross section of radius $r = R$. Eq. (1.1) implies directly that this is the only possible figure in the absence of rotation (i.e. when $\Omega = 0$).

We shall attempt to find other equilibrium figures $r = R(1 + \xi)$, $\xi \ll 1$, deviating continuously from the circular cylinder. To this end we expand (1.2) in powers of ξ, limiting ourselves to terms of the order ξ^3,

$$\xi'' + (4\omega^2 R^3 - 1 - D) + \xi(16\omega^2 R^3 - 1 - 2D) + \xi^2(24\omega^2 R^3 - D) + \xi^3 16\omega^2 R^3 +$$
$$+ \xi'^2(6\omega^2 R^3 - 2 - {}^3/_2 D) + \xi\xi'^2(12\omega^2 R^3 + 2) = 0 \tag{2.1}$$

We attempt to solve (2.1) by the usual procedure [2] of the expanding in powers of small amplitude of the deviation ξ,

$$\xi = \xi_0 + \xi_1 + \xi_2 + \ldots$$
$$D = D_* + \delta_1 + \delta_2 + \ldots \tag{2.2}$$
$$8\omega^2 R^3 = 8\omega_*^2 R^3 + \varepsilon_1 + \varepsilon_2 + \ldots$$

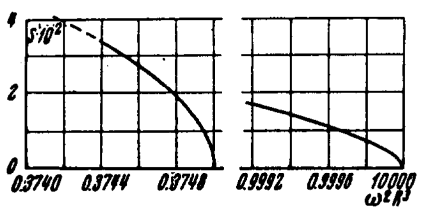

Fig. 1

Substituting (2.2) into (2.1), we obtain equations for the successive approximations. In the zeroth approximation with allowance for the condition (1.3) we obtain

$$\xi_0 = 0, \quad 4\omega_*^2R^3 - 1 - D_* = 0 \tag{2.3}$$

The equation for the first approximation is

$$\xi_1'' + (\tfrac{1}{2}\varepsilon_1 - \delta_1) + \xi_1(2D_* + 3) = 0$$

In this equation $(\tfrac{1}{2}\varepsilon_1 - \delta_1) = 0$ by virtue of (1.3). Single-valuedness condition (1.5) implies that $(2D_* + 3)$ must equal the square of the integer n. This together with (2.3) yields Eqs.

$$(\tfrac{1}{2}\varepsilon_1 - \delta_1) = 0, \quad (2D_* + 3) = n^2, \quad \omega_*^2R^3 = \tfrac{1}{8}(n^2 - 1), \quad \xi_1 = s\cos n\varphi \tag{2.4}$$

Availing ourselves of the arbitrary choice of the origin of φ, we set the initial phase in ξ_1 equal to zero. Finally, satisfying conditions (1.4), we find that $n \neq 1$, $n = 2,3,\ldots$, i.e. that a solution of the $\cos\varphi$ type does not exist because it would violate the requirement that the line of centers of mass coincide with the axis of rotation.

The expression $8\omega_*^2R^3 = n^2 - 1$ in (2.4) defines the bifurcation points, i.e. the points at which the equilibrium cylindrical figures with normal cross sections with nth order $(n = 2,3,\ldots)$-axes of symmetry branch off from the cylinders with circuar cross sections.

The equation for the second approximation is

$$\xi_2'' + n^2\xi_2 = -\xi_1(2\varepsilon_1 - 2\delta_1) + (\delta_2 - \tfrac{1}{2}\varepsilon_2) + \xi_1^2(6 + 5D) + \tfrac{1}{2}\xi_1'^2 \tag{2.5}$$

From condition (1.5) and (2.4) we find that $\varepsilon_1 = \delta_1 = 0$. The solution of Eq.(2.5) is

$$\xi_2 = \frac{1}{n^2}(\delta_2 - \tfrac{1}{2}\varepsilon_2) + \frac{s^2}{4n^2}(3 - 4n^2) + \frac{s^2}{4n^2}(2n^2 - 1)\cos 2n\varphi$$

Conditions (1.4) are fulfilled automatically, and condition (1.3) establishes the relationship among δ_2, ε_2 and s^2,

$$\delta_2 - \tfrac{1}{2}\varepsilon_2 + \tfrac{3}{4}(1 - n^2)s^2 = 0 \tag{2.6}$$

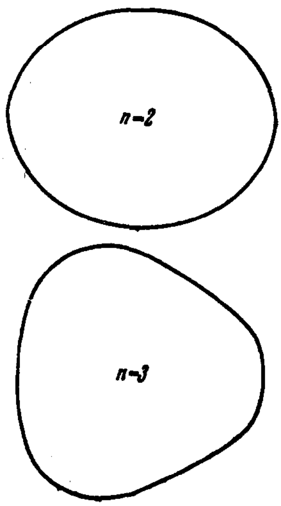

Fig. 2

One more equation for these quantities results in the third approximation upon fulfillment of periodicity condition (1.5),

$$\tfrac{1}{2}\varepsilon_2(3 - n^2) + 3\delta_2(n^2 - 1) + \tfrac{1}{8}s^2(-15 + 33n^2 - 21n^4 + 3n^6) = 0 \tag{2.7}$$

Eqs. (2.6) and (2.7) enable us to express ε_2 and δ_2 in terms of s^2. Substituting the resulting corrections ε_1 and ε_2 into (2.2), we obtain an expression relating the angular velocity and the amplitude s of the deviation of the normal cross section of the equilibrium right cylinder from circular shape in the neighborhood of the bifurcation points ω_* (Fig. 1),

$$\omega^2R^3 = \frac{n^2 - 1}{2} - \frac{3s^2}{64n^2}(1 - n^2)^2(1 + n^2) \tag{2.8}$$

Fig. 2 shows these noncircular equilibrium figures. From (2.8) we see that in this approximation the branches of the noncircular cross sections behave as quadratic parabolas which turn towards the left at the point $\omega = 0$. It is clear, however, that the curves cannot reach this point, since the only possible equilibrium figure in the absence of rotation is the circular cylinder. Hence, each curve has its own turning point. These points can be determined in the subsequent approximations.

3. The variational method. In order to construct the brances of the noncircular cross sections in the subsequent approximations we make use of the fact that the problem of equilibrium forms can be reduced to a variational problem. This is because the equilibrium figures correspond to the energy extremum under additional conditions (1.3) – (1.5). In the coordinate system rotating together with the liquid mass, the energy ε per unit length of the cylinder is given by

$$\varepsilon = -\frac{\rho\Omega^2}{8}\int_0^{2\pi} r^4\, d\varphi + \alpha \int_0^{2\pi} \sqrt{r^2 + r'^2}\, d\varphi \tag{3.1}$$

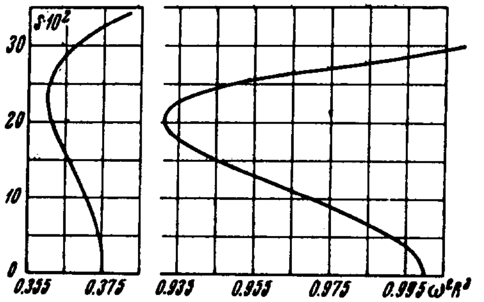

Fig. 3

We attempt to find the function $r = r(\varphi)$ by the Ritz method in a form satisfying conditions (1.4) and (1.5),

$$r(\varphi) = a(1 + s\cos n\varphi + p\cos 2n\varphi)$$

$$(n = 2, 3, \ldots) \quad \left(a^2 = \frac{2R^2}{2 + s^2 + p^2}\right) \tag{3.2}$$

The quantity a is here defined by condition (1.3), while s and p are the variational formulas with respect to which the energy is extremized.

Since substitution of (3.2) into (3.1) results in a complex expression, we expand in powers of s and p up to s^6 and p^3, inclusive, in (3.1),

$$\frac{\varepsilon}{\alpha} = -2\pi\omega^2 R^4 \left(1 + 2s^2 - \frac{15}{8}s^4 + \frac{11}{8}s^6 + 2p^2 + 3s^2 p - 3s^2p^2 - 3s^4 p\right) +$$

$$+ 2\pi R\left[1 + s^2\,\frac{n^2-1}{4} + s^4\left(\frac{3}{32} - \frac{3n^4}{64}\right) + s^6\left(-\frac{5}{128} + \frac{5n^2}{128} - \frac{9n^4}{256} + \frac{5n^6}{256}\right) +\right.$$

$$\left. + p^2\left(n^2 - \frac{1}{4}\right) + s^2 p\left(-\frac{3n^2}{8}\right) + s^2p^2\left(\frac{3}{16} + \frac{5n^2}{16} - \frac{3n^4}{4}\right) + s^4 p\left(-\frac{5n^2}{32} + \frac{9n^4}{32}\right)\right]$$

In this expression we use the same definition for ω as in (1.2). Extremizing the above equation with respect to the parameters s^2 and p, and carrying out some simple computations, we obtain

$$\omega^2 R^3 = \frac{n^2-1}{8} - \frac{3(1-n^2)^2(1+n^2)}{64n^2}s^2 +$$

$$+ (3 - 30n^2 + 79n^4 + 14n^6 - 197n^8 + 141n^{10} + 15n^{12})\frac{s^4}{512n^6} \tag{3.3}$$

The shapes of bifurcation branches (3.3) are shown in Fig. 3. As expected, these branches have turning points. In the neighborhoods of the bifurcation points, when we can limit ourselves to quadratic terms in s, these branches coincide with parabolas (2.8) obtained above.

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